

# Computing “Small” 1–Homological Models for Commutative Differential Graded Algebras

CHATA group  
Dept. of Applied Math.,  
University of Seville, Spain,  
real@us.es

## Abstract

We use homological perturbation machinery specific for the algebra category [13] to give an algorithm for computing the differential structure of a small 1–homological model for commutative differential graded algebras (briefly, CDGAs). The complexity of the procedure is studied and a computer package in Mathematica is described for determining such models.

## 1 Introduction

The description of efficient algorithms for homological computation can be considered to be a very important topic in Homological Algebra. These algorithms can be used mainly in the resolution of problems in Algebraic Topology; but this subject also impinges directly on the development of diverse areas such as Combinatorial Designs, Code Theory, Concurrency Theory or Cohomological Physics.

Starting from a finite CDGA  $A$ , we establish an algorithm for obtaining an “economical” 1–homological model  $hBA$ , in the sense that the number of algebra generators of  $hBA$  is less than that of the reduced bar construction  $\bar{B}(A)$ . In order to get the 1–homology of  $A$ , we would need to compute the homology groups of the model  $hBA$ .

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This computation can be reduced to a simple problem of Linear Algebra (see [11] for a complete explanation of this method).

Our main technique is homological perturbation machinery [6, 7, 9]. Homological Perturbation Theory is often used to replace given chain complexes by other smaller, homotopic chain complexes which are more readily computable. An essential notion in this theory is that of contraction. A contraction  $c = (f, g, \phi)$  between two differential graded modules  $(N, d_N)$  and  $(M, d_M)$  is a special homotopy equivalence between both modules such that the corresponding homology groups are isomorphic. The morphisms  $f$ ,  $g$  and  $\phi$  are called *projection*, *inclusion* and *homotopy* of the contraction, respectively. The Basic Perturbation Lemma is the heart of this theory and states that given a contraction  $c = (f, g, \phi)$ , and a perturbation  $\delta$  of  $d_N$  (that is,  $(d_N + \delta)^2 = 0$ ), then there exists a new contraction  $c_\delta = (f_\delta, g_\delta, \phi_\delta)$  from  $(N, d_N + \delta)$  to  $(M, d_M + d_\delta)$ , satisfying

$$\begin{aligned} f_\delta &= f(1 - \delta \Sigma_c^\delta \phi), & g_\delta &= \Sigma_c^\delta g, & \phi_\delta &= \Sigma_c^\delta \phi \\ d_\delta &= f \delta \Sigma_c^\delta g, \end{aligned} \tag{1}$$

where  $\Sigma_c^\delta = \sum_{i \geq 0} (-1)^i (\phi \delta)^i = 1 - \phi \delta + \phi \delta \phi \delta - \cdots + (-1)^i (\phi \delta)^i + \cdots$ .

It is necessary to emphasize that a nilpotent condition for the composition  $\delta \phi$  is required for guaranteeing the finiteness of the formulas.

The basic idea we use in this paper is the establishment (via composition, tensor product or perturbation of contractions) of an explicit contraction from an initial differential graded module  $N$  to a free differential graded module  $M$  of finite type, so that the homology of  $N$  is computable from that of  $M$ .

This “modus operandi” has been used by the authors in previous works [1, 8, 3].

Working in the context of CDGAs, Homological Perturbation Theory immediately supplies a general algorithm computing the 1-homology of these objects at graded module level. Nevertheless, this procedure, already presented by Lambe in [12], bears, in general, high computational charges and actually restricts its application to the low dimensional homological calculus.

This algorithm is refined, taking advantage of the multiplicative structures, in [2]. More precisely, the Semifull Algebra Perturbation Lemma [13, Sec. 4] is used for designing the algorithm **Alg1**. The input of this method is a CDGA  $A$  given in the form of a “twisted” tensor product of  $n$  exterior and polynomial algebras, and the output is a contraction  $c_\delta$  (produced via perturbation) from the reduced bar construction  $\bar{B}(A)$  to a smaller differential graded algebra  $hBA$ , which is free and of finite type as a graded module. In this case, we say that the pair  $\{c_\delta, hBA\}$  (or, simply,  $hBA$ ) is a 1-homological

model of  $A$ . Taking advantage of the fact that the differential  $d_{hBA}$  of  $hBA$  is a derivation, that is, a morphism compatible with the product of the 1-homological model, it is only necessary to know the value of this morphism applied to the generators of the model (let us observe that there are  $n$  algebra generators). This implies a substantial improvement in the computation of the differential on the small model  $hBA$ .

In spite of this improvement, the computational cost for determining the morphism  $d_{hBA}$  applied to an algebra generator of  $hBA$  is enormous, since the differential  $d_{hBA}$  follows the formula (1) and the homotopy  $\phi$  of  $c$  has an essentially exponential nature not only in time but also in space.

We develop some techniques, which comprise what we call *Inversion Theory* and which first appears in [13]. In consequence, we refine the formula for  $\phi$ , which is involved in the description of  $d_{hBA}$ . This study is based on the observation that the projection  $f$  applied to certain elements (those “with inversions”) is always null. It follows that a not insignificant number of terms in the formula of the morphism  $\phi$  can be eliminated in the composition  $f\delta(\phi\delta)^ig$ , which appears in the formula (1) of  $d_\delta$ . In such a way, we derive an upgraded algorithm **Alg2**.

The article is organized as follows: Notation and terminology are introduced in Section 2. In Section 3 the algorithm **Alg1**, which was described in [2] is recalled. Our contribution starts in Section 4 which is devoted to explaining Inversion Theory and describing the algorithm **Alg2**. An analysis of the complexity of **Alg1** and **Alg2** for computing the differential structure of the small 1-homological model  $hBA$  is carried out in Section 5 and a comparison between both algorithms is given. Finally, in Section 6 we also give several examples illustrating the implementation of **Alg2** carried out using Mathematica 3.0.

## 2 Preliminaries

Although relevant notions of Homological Algebra are explained through the exposition of this paper, most common concepts are not explicitly given (they can be found, for instance, in [10] or [15]).

Let  $\Lambda$  be a commutative ring with the non zero unit, which will be considered to be the ground ring. A *DGA-module*  $(M, d_M, \xi_M, \eta_M)$  is a module endowed with:

- A graduation, that is,  $M = \bigoplus_{n \in \mathbf{N}} M_n$ .
- A differential,  $d_M : M \rightarrow M$ , which decreases the degree by one and satisfies  $d_M^2 = 0$ .
- An augmentation,  $\xi_M : M \rightarrow \Lambda$ , with  $\xi_M d_1 = 0$ .

- A coaugmentation,  $\eta : \Lambda \rightarrow M$ , with  $\xi_M \eta_M = 1_\Lambda$ .

We will respect Koszul conventions. The *homology* of a differential graded module  $M$ , is a graded module  $H_*(M)$ , where  $H_n(M) = \text{Ker } d_n / \text{Im } d_{n+1}$ . We are specially interested on CDGAs,  $(A, d_A, *_A, \xi_A, \eta_A)$  which are differential graded modules endowed with a product,  $*_A$ , that is commutative in a graded sense. A morphism  $\delta : A \rightarrow A$  which decreases the degree by one, is a *derivation* if  $\delta *_A = *_A(1 \otimes \delta + \delta \otimes 1)$ .

Three particular algebras are of special interest in the development of this paper: exterior, polynomial and divided power algebras. Let  $n$  be a fixed non-negative integer.

- The *exterior algebra*  $E(x, 2n+1)$  is the graded algebra with generators 1 and  $x$  of degrees 0 and  $2n+1$ , respectively, and the trivial product, that is,  $x \cdot x = 0$  and  $x \cdot 1 = x$ .
- The *polynomial algebra*  $P(y, 2n)$  consists in the graded algebra with generators 1 of degree 0 and  $y$  of degree  $2n$ . The product is the usual one in polynomials, i.e.:  $y^i \cdot y^j = y^{i+j}$ , for non negative integers  $i$  and  $j$ .
- Finally, the *divided power algebra*  $\Gamma(y, 2n)$  is the graded algebra with generators 1 and  $y$  ( $y = y^{(1)}$ ) of respective degrees 0 and  $2n$ . The product is defined by the rules  $y^{(i)} \cdot y^{(j)} = \binom{i+j}{i} y^{(i+j)}$ ,  $i$  and  $j$  being non-negative integers.

Each one of these three types of algebras can be considered as a CDGA with the trivial differential.

Now, we shall recall a standard algebraic tool which allows us to preserve the product structure of the initial CDGA through the procedure of homological computation. The *reduced bar construction* [10] associated to a CDGA  $A$  is defined as the differential graded module  $\bar{B}(A)$ :

$$\bar{B}(A) = \Lambda \oplus \text{Ker } \xi_A \oplus (\text{Ker } \xi_A \otimes \text{Ker } \xi_A) \oplus \cdots \oplus (\text{Ker } \xi_A \otimes \cdots \otimes \text{Ker } \xi_A) \oplus \cdots .$$

An element from  $\bar{B}(A)$  is denoted by  $\bar{a} = [a_1 | \cdots | a_n]$ . There is a *tensor graduation*  $| \cdot |_t$  given by  $[a_1 | \cdots | a_n]_t = \sum_{i=1}^n |a_i|$ , as well as a *simplicial graduation*  $| \cdot |_s$ , which is defined by  $|\bar{a}|_s = |[a_1 | \cdots | a_n]|_s = n$ . The total degree of  $\bar{a}$  is given by  $|\bar{a}| = |\bar{a}|_t + |\bar{a}|_s$ .

The total differential is given by the sum of the tensor and simplicial differentials. The *tensor differential* is defined by:

$$d_t[a_1 | \cdots | a_n] = - \sum_{i=1}^n (-1)^{[a_1] \cdots [a_{i-1}]} [a_1 | \cdots | d_A a_i | \cdots | a_n] .$$

The *simplicial differential* acts by cutting down the simplicial degree by using the product given in  $A$ .

When the algebra  $A$  is commutative, it is possible to define a multiplicative structure on  $\bar{B}(A)$  (via an operator called the *shuffle product*), so that the reduced bar construction also becomes a CDGA.

Given two non-negative integers  $p$  and  $q$ , a  $(p, q)$ -*shuffle* is defined as a permutation  $\pi$  of the set  $\{0, \dots, p+q-1\}$ , such that  $\pi(i) < \pi(j)$  when  $0 \leq i < j \leq p-1$  or  $p \leq i < j \leq p+q-1$  is the case.

Let us observe that there are  $\binom{p+q}{p}$  different  $(p, q)$ -shuffles.

So, given a CDGA  $A$ , the *shuffle product*  $\star : \bar{B}(A) \otimes \bar{B}(A) \longrightarrow \bar{B}(A)$ , is defined by:

$$[a_1 | \dots | a_p] \star [b_1 | \dots | b_q] = \sum_{\pi \in \{(p,q)\text{-shuffles}\}} (-1)^{\varepsilon(\pi, a, b)} [c_{\pi(0)} | \dots | c_{\pi(p+q-1)}];$$

where  $(c_0, \dots, c_{p-1}, c_p, \dots, c_{p+q-1}) = (a_1, \dots, a_p, b_1, \dots, b_q)$  and

$$\varepsilon(\pi, a, b) = \sum_{\pi(i) > \pi(p+j)} |[a_i]| |[b_j]|.$$

Let  $n$  be a non-negative integer. The  $n$ -homology of a CDGA  $A$  (see [10]) consists in the homology groups of the iterated reduced bar construction  $\bar{B}^n(A) = \bar{B}(\bar{B}^{n-1}(A))$ , being  $\bar{B}^0(A) = A$ .

Let  $\{A_i\}_{i \in I}$  be a set of CDGAs. A *twisted tensor product*  $\tilde{\otimes}_{i \in I}^{\rho} A_i$  is a CDGA satisfying the following conditions:

- i)  $\tilde{\otimes}_{i \in I}^{\rho} A_i$  coincides with the tensor product  $\otimes_{i \in I} A_i$  as a graded algebra.
- ii) The differential operator consists in the sum of the differential of the banal tensor product and a derivation  $\rho$ .

A *contraction*  $c : \{N, M, f, g, \phi\}$  [4, 9], also denoted by  $(f, g, \phi) : N \xrightarrow{c} M$ , from a differential graded module  $(N, d_N)$  to a differential graded module  $(M, d_M)$  consists in a homotopy equivalence determined by three morphisms  $f$ ,  $g$  and  $\phi$ ;  $f : N_* \rightarrow M_*$  (projection) and  $g : M_* \rightarrow N_*$  (inclusion) being two differential graded module morphisms and  $\phi : N_* \rightarrow N_{*+1}$  a homotopy operator. Moreover, these data are required to satisfy the following rules:

$$fg = 1_M, \quad \phi d_N + d_N \phi + gf = 1_N, \quad f\phi = 0, \quad \phi g = 0, \quad \phi\phi = 0.$$

There are two basic operations between contractions which give place to new contractions: tensor product and composition of contractions.

In this paper we use a particular type of contraction between CDGAs. Given two CDGAs  $A$  and  $A'$ , a *semifull algebra contraction*  $(f, g, \phi) : A \Rightarrow A'$  [13] consists of an inclusion  $g$  that is a morphism of CDGAs, a quasi-algebra projection  $f$  and a quasi-algebra homotopy  $\phi$ . We recall that

1. The projection  $f$  is said to be a *quasi-algebra projection* whenever the following conditions hold:

$$f(\phi *_A \phi) = 0, \quad f(\phi *_A g) = 0, \quad f(g *_A \phi) = 0. \quad (2)$$

2. The homotopy operator  $\phi$  is said to be a *quasi-algebra homotopy* if

$$\phi(\phi *_A \phi) = 0, \quad \phi(\phi *_A g) = 0, \quad \phi(g *_A \phi) = 0. \quad (3)$$

The class of all semifull algebra contractions is closed under composition and tensor product of contractions. Moreover, this class is closed under perturbation.

**Theorem 2.1** [13]

Let  $c : \{N, M, f, g, \phi\}$  be a semifull algebra contraction and  $\delta : N \rightarrow N$  be a perturbation-derivation of  $d_N$ . Then, the perturbed contraction  $c_\delta$ , is a new semifull algebra contraction.

### 3 Computability of the 1-Homology of CDGAs. First Algorithm

Here we recall the algorithm described in [2] for the computation of a 1-homological model of a CDGA.

It is commonly known that every CDGA  $A$  “factors”, up to homotopy equivalence, into a tensor product of exterior and polynomial algebras endowed with a differential-derivation; in the sense that there exists a homomorphism connecting both structures, which induces an isomorphism in homology.

In fact, our input is a twisted tensor product of algebras  $A = \tilde{\otimes}_{i \in I}^\rho A_i$  where  $I$  denotes a finite set of indices,  $\rho$  is a differential-derivation and  $A_i$  an exterior or a polynomial algebra, for every  $i$ . In our algorithmic approach, we encode  $A$  by

1. a sequence of non-negative integers  $n_1 \leq n_2 \leq \dots \leq n_k$ , such that  $n_i$  represents the degree of the algebra generator  $x_i$  of  $A_i$ ;
2. a  $k$ -vector  $\bar{v} = (v_1, v_2, \dots, v_k)$ , such that  $v_i$  is  $\rho(x_i)$  for all  $i$ .

The principal goal is to obtain a “chain” of semifull algebra contractions starting at the reduced bar construction  $\bar{B}(A)$  and ending up at a smaller free (as a module) CDGA. In that way, we determine a 1-homological model for  $A$ .

Now, we consider the following three **semifull** algebra contractions which are used, firstly, to find the structure of a graded module of a 1-homological model for a CDGA:

- The contraction defined in [4, 5] from  $\bar{B}(A \otimes A')$  to  $\bar{B}(A) \otimes \bar{B}(A')$ , where  $A$  and  $A'$  are two CDGAs.

$$\begin{aligned}
& C_{\bar{B} \otimes} : \{ \bar{B}(A \otimes A'), \bar{B}(A) \otimes \bar{B}(A'), f_{\bar{B} \otimes}, g_{\bar{B} \otimes}, \phi_{\bar{B} \otimes} \}; \\
& - f_{\bar{B} \otimes} [a_1 \otimes a'_1 | \dots | a_n \otimes a'_n] \\
& = \sum_{i=0}^n \xi_A(a_{i+1} *_{A'} \dots a_n) \xi_{A'}(a'_1 *_{A'} \dots a'_i) [a_1 | \dots | a_i] \otimes [a'_{i+1} | \dots | a'_n] \\
& - g_{\bar{B} \otimes} ([a_1 | \dots | a_n] \otimes [a'_1 | \dots | a'_n]) \\
& = [a_1 \otimes \theta' | \dots | a_n \otimes \theta'] \star [\theta \otimes a'_1 | \dots | \theta \otimes a'_n],
\end{aligned}$$

where  $\theta$  and  $\theta'$  are the units in  $A$  and  $A'$  respectively.

$$\begin{aligned}
& - \text{up to sign, } \phi_{\bar{B} \otimes} ([a_1 \otimes a'_1 | \dots | a_n \otimes a'_n]) \\
& = \sum \pm \xi_A(a_{n-q+1} *_{A'} \dots a_n) [a_1 \otimes a'_1 | \dots | a_{\bar{n}-1} \otimes a'_{\bar{n}-1} \\
& \quad | a'_{\bar{n}} *_{A'} \dots a'_{n-q} | c_{\pi(0)} | \dots | c_{\pi(p+q)}],
\end{aligned}$$

where  $\bar{n} = n - p - q$ ,  $(c_0, \dots, c_{p+q}) = (a_{\bar{n}}, \dots, a_{n-q}, a'_{n-q+1}, \dots, a'_n)$  and the sum is taken over all the  $(p+1, q)$ -shuffles  $\pi$  and  $0 \leq p \leq n - q - 1 \leq n - 1$ .

Let us note that the complexity of  $g_{\bar{B} \otimes}$  and  $\phi_{\bar{B} \otimes}$  is exponential since shuffles are involved in both formulas.

Given a tensor product  $\otimes_{i \in I} A_i$  of CDGAs, a contraction from  $\bar{B}(\otimes_{i \in I} A_i)$  to  $\otimes_{i \in I} \bar{B}(A_i)$  is easily determined by applying  $C_{\bar{B} \otimes}$  several times in a suitable way. This new contraction is also denoted by  $C_{\bar{B} \otimes}$ .

- The isomorphism of differential graded algebras (therefore, a contraction)

$$C_{\bar{B}E} : \{\bar{B}(E(u, 2n+1)), \Gamma(\underline{u}, 2n+2), f_{\bar{B}E}, g_{\bar{B}E}, 0\}$$

described in [5], where

$$f_{\bar{B}E}([u | \overset{m \text{ times}}{\cdots} | u]) = \underline{u}^{(m)}; \quad g_{\bar{B}E}(\underline{u}^{(m)}) = [u | \overset{m \text{ times}}{\cdots} | u].$$

- The contraction

$$C_{\bar{B}P} : \{\bar{B}(P(v, 2n)), E(\underline{v}, 2n+1), f_{\bar{B}P}, g_{\bar{B}P}, \phi_{\bar{B}P}\}$$

stated in [5], where

$$f_{\bar{B}P}([v^r]) = \begin{cases} 0 & \text{if } r \neq 1 \\ \underline{v} & \text{if } r = 1 \end{cases}, \quad f_{\bar{B}P}([v^{r_1} | \cdots | v^{r_m}]) = 0;$$

$$g_{\bar{B}P}(\underline{v}) = [v] \quad \text{and} \quad \phi_{\bar{B}P}([v^{r_1} | \cdots | v^{r_m}]) = [v | v^{r_1-1} | \cdots | v^{r_m}].$$

Thanks to these three contractions, it is possible to establish, by composition and tensor product of contractions, the following semifull algebra contraction  $C = (f, g, \phi)$ :

$$\bar{B}(\otimes_{i \in I} A_i) \Rightarrow \otimes_{i \in I} \bar{B}(A_i) \Rightarrow \otimes_{i \in I} hBA_i,$$

where  $hBA_i$  represents an exterior or a divided power algebra with a generator  $\underline{x}_i$ , depending on whether  $A_i$  is a polynomial or an exterior algebra with a generator  $x_i$ .

In order to obtain the differential structure of the 1-homological model for the twisted tensor product  $\tilde{\otimes}_{i \in I}^\rho A_i$ , the next step is to perturb  $C$ . The perturbation  $\rho$  produces a perturbation-derivation  $\delta$  on the tensor differential of  $\bar{B}(\otimes_{i \in I} A_i)$ :

$$\delta([a_1 | \cdots | a_n]) = \sum_{i=1}^n (-1)^{|[a_1 | \cdots | a_{i-1}]|} [\rho(a_i) | \cdots | a_n].$$

Now, by applying Theorem 2.1, a new semifull algebra contraction  $(f_\delta, g_\delta, \phi_\delta)$  is constructed:

$$\bar{B}(\tilde{\otimes}_{i \in I}^\rho A_i) \xrightarrow{(C)_\delta} (\otimes_{i \in I} hBA_i, d_\delta),$$

where the differential  $d_\delta$  is determined by the perturbation procedure (Basic Perturbation Lemma). That means that  $hBA = \otimes_{i \in I} (hBA_i, d_\delta)$  is a 1-**homological model** of  $A = \tilde{\otimes}_{i \in I} A_i$ . Let us emphasize that the Basic Perturbation Lemma provides finite formulas. Indeed, this is a consequence of two facts: the perturbation  $\delta$  does not change the simplicial degree and  $\phi$  increases this degree.



**Procedure 1** *Algorithm Alg1.*

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**Input:**  $A$  finite CDGA  $A: ((n_1, \dots, n_k), (v_1, \dots, v_k))$ .

**Output:**  $((n_1 + 1, \dots, n_k + 1), (w_1, \dots, w_k))$   
a 1-homological model of the CDGA  $\otimes_{i=1, \dots, k}^\rho A_i$ ,  $A_i$  being the  
exterior algebra  $E(x_i, n_i)$ , if  $n_i$  is odd and  $P(x_i, n_i)$  if  $n_i$  is even.

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$w_1 = 0$ ,

**for**  $i = 2$  **to**  $k$

$w_i = d_\delta(\underline{x}_i)$ , where  $x_i$  is the algebra generator of degree  $n_i$

**endfor**

Naturally, the first components of the vector  $\bar{v}$  must be zero, because they correspond to the image of the algebra generators with the lowest degree under  $\rho$ .

Moreover, a general algorithm for computing the 1-homology of CDGAs can be described. Clearly, the homology of the 1-homological model obtained can be computed using an algorithm based on the establishment of Smith's normal form of the matrices representing the differentials at each degree [14, 11].

The computational cost of constructing the contraction  $(C)_\delta$  is high. Let us note that both the inclusion and homotopy operators of the contraction  $C_{\bar{B} \otimes}$  give an answer in exponential time. In fact, the formula of the differential operator  $d_\delta$  produced by the homological perturbation machinery is given by:

$$d_\delta = f \delta (1 - \phi \delta + \phi \delta \phi \delta - \dots) g.$$

With regard to the previous remarks, a first impression is that obtaining  $d_\delta$  generally becomes a procedure of exponential nature.

It is possible to take advantage of  $d_\delta$  being a derivation. Indeed, the fact that  $d_\delta$  is a derivation implies that it is only necessary to know this morphism applied to the generators of the model (let us observe that there are as many generators as the cardinal of the set of indices  $I$  indicated). This is an enormous improvement in the computation of the differential on the small model. In spite of this, computing  $d_\delta$  on an algebra generator is extremely time-consuming.

## 4 Inversion Theory

In this section, we go further in the simplification of the computation of the differential  $d_\delta$ . For clarity, we begin this work considering only two algebras.

As we have seen before, obtaining  $d_\delta$  is an extremely expensive procedure. The morphism responsible for this is the homotopy operator,  $\phi$ , due, essentially, to the shuffles that are involved in the formulas of  $\phi_{B\otimes}$  and  $g_{B\otimes}$ . We intend to eliminate these shuffles, and, with this aim in mind, we define the concept of inversion.

**Definition 4.1** Let  $A$  and  $A'$  be CDGAs and let us consider a homogeneous element  $[a_1 \otimes a'_1 | a_2 \otimes a'_2 | \cdots | a_n \otimes a'_n]$  from  $\bar{B}(A \otimes A')$ . We say that a component  $\theta \otimes a'_i$  from that element, is responsible for an *inversion*, if there exists an index  $j > i$  with  $a_j \neq \theta$  (where  $\theta$  is the unit of  $A$ ). In this sense, such an element presents  $k$  inversions if there exist  $k$  components responsible for an inversion.

We will say that an element from  $\bar{B}(A \otimes A')$  has  $k$  inversions, if it is a sum of elements which each have, as a minimum,  $k$  inversions.

Let us consider the contraction

$$(f_{\bar{B}\otimes}, g_{\bar{B}\otimes}, \phi_{\bar{B}\otimes}) : \bar{B}(A \otimes A') \Rightarrow \bar{B}(A) \otimes \bar{B}(A')$$

described in the previous section. We analyze the behaviour of the component morphisms of this contraction with respect to inversions. For this purpose, we do not take into account the signs in the formulas referred to.

- The image of an element with at least one inversion under  $f_{\bar{B}\otimes}$ , is null.
- The injection  $g_{\bar{B}\otimes}$ , applied to  $[a_1 | \cdots | a_n] \otimes [a'_1 | \cdots | a'_m]$ , produces:
  - a unique term with no inversions (that one which comes from juxtaposition),
  - $n$  terms with one inversion,
  - $\binom{n+m}{n} - n - 1$  terms with more than one inversion.
- As for the homotopy operator  $\phi_{\bar{B}\otimes}$ , we can state that the image of a homogenous element under  $\phi_{\bar{B}\otimes}$  gives rise to a sum of elements which, if non null, have at least one more inversion than the original one. Let us note that an inversion is produced by the component  $a'_n *_{A'} \cdots a'_{n-q}$ , which is always on the left side of those components  $a_{\bar{n}}, \dots, a_{n-q}$  of each summand in the formula of  $\phi_{\bar{B}\otimes}$ .

Let us consider the contraction which provides us with a 1-homological model for the tensor product of two CDGAs,  $A$  and  $A'$ :

$$(f, g, \phi) : \bar{B}(A \otimes A') \Rightarrow \bar{B}(A) \otimes \bar{B}(A') \Rightarrow hBA \otimes hBA' \quad (4)$$

where

$$\begin{aligned} f &= (f_{\bar{B}A} \otimes f_{\bar{B}A'}) f_{\bar{B}\otimes}, \\ g &= g_{\bar{B}\otimes} (g_{\bar{B}A} \otimes g_{\bar{B}A'}) \\ \phi &= \phi_{\bar{B}\otimes} + g_{\bar{B}\otimes} (\phi_{\bar{B}A} \otimes g_{\bar{B}A'} f_{\bar{B}A'} + 1_{\bar{B}A} \otimes \phi_{\bar{B}A'}) f_{\bar{B}\otimes} \end{aligned}$$

Let us note that the image of an element with an inversion under  $f$  is also null, since the first morphism applied is  $f_{\bar{B}\otimes}$ .

Now we assume that there is a perturbation  $\rho$  of the tensor product of the algebras  $A$  and  $A'$ . This perturbation induces, in a natural way, a perturbation  $\delta$  on  $\bar{B}(A \otimes A')$ . Let us analyze the behaviour of such a morphism with respect to inversions.

**Lemma 4.2** *Let us consider a perturbation  $\delta$  for  $\bar{B}(A \otimes A')$  induced by a perturbation-derivation  $\rho$  for  $A \otimes A'$  such that  $\rho(A) \subset A$ . The image of a homogeneous element with  $k$  inversions under  $\delta$ , is a sum of elements with at least  $k - 1$  inversions.*

**Proof.**

Let us point out that a component of a homogeneous element from  $\bar{B}(A \otimes A')$  is responsible for, at most, one inversion and that  $\delta$  acts only on a component of the element at each term of the resultant sum.

□

Attending to the Basic Perturbation Lemma, one can obtain from the contraction (4), a new contraction:

$$(f_\delta, g_\delta, \phi_\delta) : \bar{B}(A \tilde{\otimes}^\rho A') \Rightarrow (hBA \otimes hBA', d_\delta).$$

We recall the formula for  $d_\delta$ :

$$d_\delta = f \delta (1 - \phi \delta + \phi \delta \phi \delta - \dots) g.$$

We can observe that  $f$  is the last morphism applied in the formula. If at any stage, an element  $y$  obtained by applying  $\phi$ , has more than one inversion, then  $\delta(y)$  will have at least one inversion. In this way, each time we apply  $\delta \phi$ , we obtain a sum of homogeneous elements with at least one inversion, and, therefore, the image of these elements under  $f$  is null. This means that we only have to consider the summands of  $\phi$  having, at most, one inversion.

In consequence, we can establish the following theorem where we considerably reduce the complexity of the computation of  $d_\delta$ .

**Theorem 4.3** *The formula for  $\phi$ , that is involved in the definition of  $d_\delta$ , is the following:*

$$\phi = \bar{\phi}_{\bar{B} \otimes} + \bar{g}_{\bar{B} \otimes}(\phi_{\bar{B} A} \otimes g_{\bar{B} A'} f_{\bar{B} A'} + 1 \otimes \phi_{\bar{B} A'}) f_{\bar{B} \otimes},$$

where

$$\begin{aligned} & \bullet \bar{\phi}_{\bar{B} \otimes}([a_1 \otimes a'_1 | \cdots | a_n \otimes a'_n]) \\ &= \sum_{0 \leq p \leq n-q-1 \leq n-1} (-1)^{\varphi(n,p,q)} \xi_A(a_{n-q+1} *_A \cdots a_n) [a_1 \otimes a'_1 | \cdots | a_{\bar{n}-1} \otimes a'_{\bar{n}-1} \\ & \quad | a'_{\bar{n}} *_A \cdots a'_{n-q} | a_{\bar{n}} | \cdots | a_{n-q} | a'_{n-q+1} | \cdots | a'_n] \end{aligned}$$

being  $\bar{n} = n - p - q$  and

$$\begin{aligned} \varphi(n, p, q) &= \bar{n} - 1 + |[a_1 | \cdots | a_{\bar{n}-1}]|_t + |[a'_1 | \cdots | a'_{n-q}]|_t \\ & \quad + \sum_{k=0}^p \sum_{\ell=0}^k |a_{n-q-k}| |a'_{n-q-\ell}|. \end{aligned}$$

$$\begin{aligned} & \bullet \bar{g}_{\bar{B} \otimes}([a_1 | \cdots | a_n] \otimes [a'_1 | \cdots | a'_m]) \\ &= [a_1 | \cdots | a_n | a'_1 | \cdots | a'_m] \\ & \quad + \sum_{i=1}^{n-1} (-1)^{[a_{i+1} | \cdots | a_n] | a'_1} [a_1 | \cdots | a_i | a'_1 | a_{i+1} | \cdots | a_n | a'_2 | \cdots | a'_m] \\ & \quad + (-1)^{a'_1 | [a_1 | \cdots | a_n]} [a'_1 | a_1 | \cdots | a_n | a'_2 | \cdots | a'_m]. \end{aligned}$$

Let us note that now the number of summands in the formula above for  $\phi_{\bar{B} \otimes}$  is

$$\sum_{q=0}^{n-1} \sum_{p=0}^{n-q-1} 1 = \frac{n^2 + n}{2},$$

in contrast to the original number of summands:

$$\sum_{q=0}^{n-1} \sum_{p=0}^{n-q-1} \binom{p+q+1}{q} = 2^{n+1} - n - 2.$$

On the other hand, the formula for  $g_{\bar{B} \otimes}$  is reduced to  $n$  summands, instead of  $\binom{m+n}{n}$ .

This theorem is easy to generalize, by induction, to the general case of a twisted tensor product of CDGAs  $\tilde{\otimes}_{i \in I}^{\rho} A_i$  with  $I = \{1, \dots, n\}$ , where  $A_i$  is an exterior or a polynomial algebra with generator  $x_i$  and  $|x_i| \leq |x_{i+1}|$ . Therefore, as we saw in Section 3, the following semifull algebra contraction can be established:

$$\bar{B}(\otimes_{i=1}^n A_i) \Rightarrow \otimes_{i=1}^n \bar{B}(A_i) \Rightarrow \otimes_{i=1}^n hBA_i$$

where  $hBA_i$  is a polynomial or a divided power algebra.

The key to understanding the generalization is the fact that the inversions in  $\bar{B}(\otimes_{i=1}^n A_i)$  are those of the last tensor product (as they were defined at the beginning of the Section, with  $A = \otimes_{i=1}^{n-1} A_i$  and  $A' = A_n$ ) along with those of  $\otimes_{i=1}^{n-1} A_i = (\otimes_{i=1}^{n-2} A_i) \otimes A_{n-1}$  with respect to the last tensor product, and so on.

Summing up, we obtain an algorithm **Alg2** having the same input and output as the Algorithm **Alg1** of the last section, but speeding up the steps concerning the image of the algebra generators under  $d_{\delta}$ .

**Procedure 2** *Algorithm Alg2.*

---

**Input and Output:** *the same as in Alg1.*

---

```

 $w_1 = 0,$ 
for  $i = 2$  to  $k$ 
     $w_i = d_{\delta}(\underline{x}_i)$ , where  $x_i$  is the algebra generator of degree  $n_i$ 
    (using Theorem 4)
endfor
```

## 5 Complexity

In this section we give a comparison of the algorithms **Alg1** and **Alg2** from the point of view of their complexity. We are mainly interested in measuring the efficiency of the cor-

Table 1: Required time

		in <b>Alg1</b>	in <b>Alg2</b>
$s = 3;$	$r = 2$	0.1 sec.	0.002 sec.
$s = 3;$	$r = 3$	0.5 sec.	0.009 sec.
$s = 4;$	$r = 2$	31.29 sec.	0.04 sec.
$s = 5;$	$r = 3$	3.19 days	17.6 sec.
$s = 6;$	$r = 2$	1.45 years	1.17 min
$s = 6;$	$r = 3$	24.75 years	19.56 min

responding steps concerning the obtention of the differential  $d_\delta$ . We consider the degree of the algebra generator as the size of an instance. We take as elementary operations those ones generating each homogeneous term produced by the different morphisms and a worst-case analysis of the algorithms is carried out.

We calculate the total number of elementary operations needed for computing  $d_\delta$  on a generator  $\underline{x}_k$  of degree  $k$ , for both **Alg1** and **Alg2**. We hold that this number for **Alg1** is

$$\sum_{i=0}^{\lfloor k/k_0+1 \rfloor} (i! r^i + ((i+1)! + (i+2)!) r^{i+1}) \left( \prod_{j=1}^i 2^{j+1} - j - 2 \right),$$

and for **Alg2** is

$$\sum_{i=0}^{\lfloor k/k_0+1 \rfloor} (r^i + (i+3)r^{i+1}) \left( \prod_{j=1}^i \frac{j^2 + j}{2} \right),$$

where  $k_0 = \min_{1 \leq i \leq n} |x_i|$  and  $r$  is the maximum number of summands given by  $\rho(x_i)$ , where  $x_i$  ranges over all the algebra generators of  $A$ .

In the following table, the required time for computing  $d_\delta(\underline{x}_k)$  is showed, supposing that our computer carries out  $10^6$  elementary operations per second. Let us denote  $s = \lfloor k/k_0 + 1 \rfloor$ . Note that, for example, that  $s = 5$  means that if  $k_0 = 10$ , the degrees of the algebra generators range over the set  $\{10, 11, \dots, 54, 55\}$ .

## 6 Implementation Performance

The algorithm **Alg2** has been implemented. The user supplies an encoding of a finite CDGA in the form of a twisted tensor product of exterior and polynomial algebras to the program, which computes an encoding of a small 1-homological model of this algebra.

This program is written in Mathematica 3.0, consisting in 300 lines of code and 10 basic functions.

In order to give some indication of the implementation, we report on the time taken to compute 1-homological models for certain CDGAs:

$$1. E(x_1, x_2, x_3; 1) \otimes P(x_4, x_5; 2) \otimes E(x_6; 3) \otimes P(x_7; 4) \otimes P(x_8; 6).$$

**Input:**  $((1, 1, 1, 2, 2, 3, 4, 6),$   
 $(0, 0, 0, x_1 - x_2, x_2, x_1x_2, x_2x_4 + x_1x_5 + x_1x_2x_3, x_1x_2x_6)).$

**Output:**  $((2, 2, 2, 3, 3, 4, 5, 7), (0, 0, 0, \underline{x}_1 - \underline{x}_2, \underline{x}_2, 0, \underline{x}_1\underline{x}_2 - 2\underline{x}_2^2, 0))$

**Time:**  $d_\delta(\underline{x}_4)$  in 0.55 sec.  
 $d_\delta(\underline{x}_5)$  in 0.27 sec.  
 $d_\delta(\underline{x}_6)$  in 0.33 sec.  
 $d_\delta(\underline{x}_7)$  in 2.14 sec.  
 $d_\delta(\underline{x}_8)$  in 0.28 sec.

$$2. E(x_1; 1) \otimes P(x_2; 2) \otimes P(x_3; 6) \otimes P(x_4; 10) \otimes P(x_5; 26).$$

**Input:**  $((1, 2, 6, 10, 26), (0, -2x_1, x_1x_2^2, 3x_1x_2x_3, 8x_1x_2x_3^2x_4)).$

**Output:**  $((2, 3, 7, 11, 27), (0, -2\underline{x}_1, -8\underline{x}_1^3, -192\underline{x}_1^5, 21799895040\underline{x}_1^{13}))$

**Time:**  $d_\delta(\underline{x}_2)$  in 0.16 sec.  
 $d_\delta(\underline{x}_3)$  in 0.60 sec.  
 $d_\delta(\underline{x}_4)$  in 2.36 sec.  
 $d_\delta(\underline{x}_5)$  in 1571.47 sec.

$$3. E(x_1; 1) \otimes P(x_2; 2) \otimes P(x_3; 4) \otimes E(x_4; 5) \otimes E(x_5; 7) \otimes P(x_6; 14).$$

**Input:**  $((1, 2, 4, 5, 7, 14), (0, -x_1, x_1x_2, 0, 2x_1x_4, -x_1x_4x_5)).$

**Output:**  $((2, 3, 5, 6, 8, 15), (0, -\underline{x}_1, -\underline{x}_1^2, 0, 0))$

**Time:**  $d_\delta(\underline{x}_2)$  in 0.17 sec.  
 $d_\delta(\underline{x}_3)$  in 0.33 sec.  
 $d_\delta(\underline{x}_4)$  in 0.26 sec.  
 $d_\delta(\underline{x}_5)$  in 0.17 sec.

For each algebra, we summarize the results and the time taken to compute a description of the model. All CPU times are in seconds and calculations were carried out on a Pentium III, 128Mb RAM, 7.2Gb Hard disk space.

This program produces as output an encoding of a certain differential graded algebra which could be introduced into another program in order to calculate the homology of such objects. We intend to tackle this task in the near future.

## References

- [1] Álvarez V., Armario J.A., González-Díaz R., Real P.: *Algorithms in Algebraic Topology and Homological Algebra. The Problem of the Complexity*. Extended Abstracts of the First Workshop on Computer Algebra and Scientific Computing, CASC'98 (1998) Saint Petersburg.
- [2] Álvarez V., Armario J.A., Real P., Silva B.: *HPT and Computability of the Homology of CDGAs*. Conference on Secondary Calculus and Cohomological Physics (1997) Moscow. Electronic Proceedings of the EMS (<http://www.emis.de/proceedings>).
- [3] Armario J.A., Real P., Silva B.: *On  $p$ -Minimal Homological Models of Twisted Tensor Products of Elementary Complexes Localized over a Prime*. Contemporary Mathematics **227** (1999) 303–314.
- [4] Eilenberg S., Mac Lane S.: *On the Groups  $H(\pi, n)$ , I*. Annals of Math. **58** (1953) 55–139.
- [5] Eilenberg S., MacLane S.: *On the Groups  $H(\pi, n)$ , II*. Annals of Math. **60** (1954) 49–139.
- [6] Gugenheim V.K.A.M., Lambe L.A.: *Perturbation Theory in Differential Homological Algebra, I*. Illinois J. Math. **33** (1989) 556–582.
- [7] Gugenheim V.K.A.M., Lambe L.A., Stasheff J.D.: *Perturbation Theory in Differential Homological Algebra, II*. Illinois J. Math. **35** n. 3 (1991) 357–373.
- [8] González-Díaz R., Real P.: *Computing Cocycles on Simplicial Complexes*. Proceedings of the Second Workshop on Computer Algebra and Scientific Computing, CASC'99 (1999) Springer-Verlag, 177–190.
- [9] Huebschmann J., Kadeishvili T.: *Small Models for Chain Algebras*. Math. Zeit. **207** (1991) 245–280.
- [10] MacLane S.: *Homology*. Classics in Mathematics (1995) Springer.
- [11] Munkres, J.R.: *Elements of Algebraic Topology*. Addison-Wesley Publishing Company (1984).



- [12] Lambe L.A.: *Homological Perturbation Theory, Hochschild Homology and Formal Groups*. Contemp. Math. **134** (1992) 183–218.
- [13] Real P.: *Homological Perturbation Theory and Associativity*. To appear in Homology, Homotopy and Applications.
- [14] Veblen O.: *Analysis Situs*. A.M.S. Publications **5** (1931).
- [15] Weibel C.A.: *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press (1994).